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Noncommutative chiral anomaly and the Dirac-Ginsparg-Wilson operator

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Abstract: It is shown that the local axial anomaly in 2−dimensions emerges naturally if one postulates an underlying noncommutative fuzzy structure of spacetime. In particular the Dirac-Ginsparg-Wilson relation on $\mathbf{S}^2_F$ is shown to contain an edge effect which corresponds precisely to the "fuzzy" $\mathrm{U}(1)_A$ axial anomaly on the fuzzy sphere. We also derive a novel gauge-covariant expansion of the quark propagator in the form $\frac{1}{\mathcal{D}_{AF}}=\frac{a\hat{\Gamma}^L}{2}+\frac{1}{\mathcal{D}_{Aa}}$ where $a=\frac{2}{2l+1}$ is the lattice spacing on $\mathbf{S}^2_F$, $\hat{\Gamma}^L$ is the covariant noncommutative chirality and $\mathcal{D}_{Aa}$ is an effective Dirac operator which has essentially the same IR spectrum as $\mathcal{D}_{AF}$ but differes from it on the UV modes. Most remarkably is the fact that both operators share the same limit and thus the above covariant expansion is not available in the continuum theory. The first bit in this expansion $\frac{a\hat{\Gamma}^L}{2}$ although it vanishes as it stands in the continuum limit, its contribution to the anomaly is exactly the canonical theta term. The contribution of the propagator $\frac{1}{\mathcal{D}_{Aa}}$ is on the other hand equal to the toplogical Chern-Simons action which in two dimensions vanishes identically.

Keywords: Anomalies in Field and String Theories, Non-Commutative Geometry, Matrix Models.

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1. Introduction

Fuzzy physics ([1, 2], see also [3] and references therein), like lattice gauge theories, is aiming for a nonperturbative regularization of chiral gauge theories. Discretization in fuzzy physics is achieved by treating the underlying spacetimes as phase spaces then quantizing them in a canonical fashion which means in particular that we are effectively replacing the underlying spacetimes by non-commutative matrix models or fuzzy manifolds [3]. As a consequence, this regularization will preserve all symmetries and topological features of the problem . Indeed a fuzzy space is by construction a discrete lattice-like structure which serves to regularize, it allows for an exact chiral invariance to be formulated, but still the fermion-doubling problem is completely avoided [4].

Global chiral anomalies on these models were, along with other topological non-trivial field configurations, formulated in [5]–[8], while local anomaly on fuzzy $\mathbf{S}_F^2$ was treated first in [9] then in [10, 11]. The relevance of Ginsparg-Wilson relations in noncommutative matrix models was noted first in [4] and [12] then in [10]. In this article we will show that despite the fact that the concept of point is lacking on fuzzy $\mathbf{S}^2$, we can go beyond global considerations and define a "fuzzy" axial anomaly associated with a "fuzzy" U(1) global chiral symmetry.

The plan of the paper is as follows. Section 2 contains a brief description of fuzzy $\mathbf{S}^2$ and its star product. Fuzzy U(1) gauge theory and fermion action on $\mathbf{S}^2_F$ are introduced in section 3 where we also show the absence of the fermion doubling problem on $\mathbf{S}^2_F$. The Dirac-Ginsparg-Wilson relation on $\mathbf{S}^2_F$ and the corresponding fuzzy chiral transformations as well as the "fuzzy" $\mathrm{U}(1)_A$ axial anomaly are discussed in section 4. In section 5 we derive a novel gauge-covariant expansion of the quark propagator and show explicitly that no analogous expansion exists in the continuum theory. The continuum limit is computed in this section where beside the canonical theta term we obtain the Chern-Simons action which vanishes identically in two dimensions. We conclude in section 6.

2. The fuzzy sphere

We define the noncommutative fuzzy sphere by Connes spectral triple $(Mat_{2l+1}, \mathbf{H}_l, \Delta^F)$ [13]. The matrix algebra Mat_{2l+1} is the algebra $\mathbf{A}$ of $(2l+1)\times(2l+1)$ matrices which acts on the $(2l+1)-$dimensional Hilbert space $\mathbf{H}_l$ of the IRR l of SU(2). Matrix coordinates on $\mathbf{S}^2_F$ are defined by [1]–[3]

$$\left(n_1^F\right)^2 + \left(n_2^F\right)^2 + \left(n_3^F\right)^2 = 1\,, \qquad [n_a^F, n_a^F] = \frac{i}{\sqrt{l(l+1)}}\epsilon_{abc}n_c^F\,, \tag{2.1}$$

with

$$n_a^F = \frac{L_a}{\sqrt{l(l+1)}}\,. \tag{2.2}$$

A "fuzzy" function on $\mathbf{S}^2_F$ is a linear operator $\phi^F \in Mat_{2l+1}$. Derivations on $\mathbf{S}^2_F$ are on the other hand defined by the generators of the adjoint action of SU(2) , in other words the derivative of the fuzzy function ϕ^F in the space-time direction a is the commutator $[L_a, \phi^F]$. This can also be put in the form

$$AdL_a(\phi^F) \equiv [L_a, \phi^F] = (L_a^L - L_a^R)(\phi^F)\,, \tag{2.3}$$

where L_a^L's and $-L_a^R$'s are the generators of the IRR l of SU(2) which act respectively on the left and on the right of the algebra Mat_{2l+1}, i.e. $L_a^L\phi^F \equiv L_a\phi^F$, $L_a^R\phi^F \equiv \phi^F L_a$ for any $\phi^F \in Mat_{2l+1}$. Hence the laplacian operator Δ^F on the fuzzy sphere is simply given by the Casimir operator

$$\Delta^F = (L_a^L - L_a^R)^2\,. \tag{2.4}$$

The algebra of matrices Mat_{2l+1} decomposes therefore under the action of the group SU(2) as $l\otimes l = 0\oplus 1\oplus 2\oplus \cdots \oplus 2l$ (The first l stands for the left action of the group while the other l stands for the right action). As a consequence a general scalar function on $\mathbf{S}^2_F$ can be expanded in terms of polarization tensors as follows

$$\phi^F = \sum_{k=0}^{2l}\sum_{m=-k}^{k} \phi_{km}\hat{Y}_{km}(l)\,. \tag{2.5}$$

(For an extensive list of the properties of $\hat{Y}_{km}(l)$'s see [14]). The fact that the summation over k involves only angular momenta which are $\leq 2l$ originates from the fact that the spectrum $k(k+1)$ of the laplacian Δ^F is cut-off at $k=2l$. As we will show in this article this rotationally-invariant cut-off is non-trivial in the sense that it respects both gauge and chiral symmetries.

As one can already notice all these definitions are in very close analogy with the case of continuum $\mathbf{S}^2$ where the algebra of functions $\mathcal{A}$ plays there the same role played here by the matrix algebra Mat_{2l+1}. In fact the continuum limit is defined by $l\longrightarrow\infty$ where the fuzzy coordinates n_a^F's approach the ordinary coordinates n_a's and where the algebra Mat_{2l+1} tends to the algebra $\mathcal{A}$ in the sense that

$$\phi^F\longrightarrow\phi(\vec{n})=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\phi_{km}Y_{km}(\vec{n})\,. \tag{2.6}$$

In above $Y_{km}(\vec{n})$ stands for the canonical spherical harmonics. Correspondingly fuzzy derivations reduce to ordinary ones on commutative $\mathbf{S}^2$, i.e. $adL_a(\phi^F)\longrightarrow\mathcal{L}_a(\phi)(\vec{n})$, $\mathcal{L}_a=-i\epsilon_{abc}n_b\partial_c$. Formally one writes $\mathcal{A}=Mat_{\infty}$ and think of the fuzzy sphere as having a finite number of points equal to $2l+1$ which will diverge in the continuum limit $l\longrightarrow\infty$.

To make the continuum limit more precise we will need to introduce the star product on $\mathbf{S}^2_F$. To any fuzzy scalar function ϕ^F on $\mathbf{S}^2_F$, i.e. an operator ϕ^F acting on $\mathbf{H}_l$, we associate a "classical" function $\langle\phi^F\rangle(\vec{n})$ on a classical $\mathbf{S}^2$ by $\langle\phi^F\rangle(\vec{n})=\langle\vec{n},l|\phi^F|\vec{n},l\rangle$. The product of any two operators ϕ_1^F and ϕ_2^F is then mapped to the star product of the corresponding two functions as follows $\langle\phi_1^F\rangle*\langle\phi_2^F\rangle(\vec{n})=\langle\vec{n},l|\phi_1^F\phi_2^F|\vec{n},l\rangle$. $|\vec{n},l\rangle$ is of course the canonical SU(2) coherent state in the IRR l. A long calculation shows that this star product is given by [15]

$$\begin{aligned}\langle\phi_1^F\rangle*\langle\phi_2^F\rangle(\vec{n})&=\sum_{k=0}^{2l}\frac{(2l-k)!}{k!(2l)!}K_{a_1b_1}\cdots K_{a_kb_k}\frac{\partial}{\partial n^{a_1}}\cdots\frac{\partial}{\partial n^{a_k}}\langle\phi_1^F\rangle(\vec{n})\frac{\partial}{\partial n^{b_1}}\cdots\frac{\partial}{\partial n^{b_k}}\langle\phi_2^F\rangle(\vec{n})\\ K_{ab}&=\delta_{ab}-n_an_b-i\epsilon_{abc}n_c\,.\end{aligned} \tag{2.7}$$

In these coherent states one can also compute

$$\langle n_a^F\rangle=\frac{1}{\sqrt{1+1/l}}n_a\,,\qquad\langle[L_a,\phi^F]\rangle=(\mathcal{L}_a\langle\phi^F\rangle)(\vec{n})\,, \tag{2.8}$$

and

$$\frac{1}{2l+1}\,\mathrm{Tr}_l\,\phi_1^F\phi_2^F=\int_{\mathbf{S}^2}\frac{d\Omega}{4\pi}\langle\phi_1^F\rangle*\langle\phi_2^F\rangle(\vec{n})\,. \tag{2.9}$$

The trace Tr_l is obviously taken over the Hilbert space $\mathbf{H}_l$. Remark finally that the coherent state $|\vec{n},l\rangle$ becomes localized around the point $\vec{n}$ in the limit and as a consequence $\mathrm{Lim}\langle n_a^F\rangle=n_a$, $\mathrm{Lim}\langle\phi^F\rangle=\phi$ and $\mathrm{Lim}\langle[L_a,\phi_a]\rangle=\mathcal{L}_a\phi$, etc. In this limit the star product reduces also to the ordinary product of functions.

3. Fuzzy actions

3.1 Gauge fields on $\mathbf{S}^2_F$

Next we would like to write down the Schwinger model on $\mathbf{S}^2_F$. First we introudce 2−dimensional gauge fields on the fuzzy sphere and their action. A vector field $\vec{A}^F$ on the fuzzy sphere is an element of the projective module $Mat_{2l+1}\otimes\mathbf{C}^3$. Writing a gauge principle for this matrix vector field is not difficult, indeed the action takes the usual form

$$S_{\rm YMF}=\frac{1}{4e^2}\frac{1}{2l+1}\,{\rm Tr}_l\,F^F_{ab}F^F_{ab}\,. \tag{3.1}$$

The curvature $F^F_{ab}=-F^{F+}_{ab}$ is also given by the usual formula $F^F_{ab}\equiv[D^F_a,D^F_b]-i\epsilon_{abc}D^F_c$ with the covariant derivative $D^F_a=L_a+A^F_a$. U(1) gauge transformations are implemented by unitary transformations acting on the $(2l+1)-$dimensional Hilbert space of the irreducible representation l of SU(2), namely $D^F_a\longrightarrow D^{F'}_a=U^FD^F_aU^{F+}$ where $U^F=e^{i\Omega^F}$ and $\Omega^F=\Omega^{F+}$ is an element of the algebra Mat_{2l+1}.

Using the rule of the star product it is not difficult to convince ourselves that (3.1) has the correct continuum limit, i.e. $S_{\rm YMF}\longrightarrow S_{YM}=\frac{1}{4e^2}\int_{{\bf S}^2}\frac{d\Omega}{4\pi}F_{ab}F_{ab}$, $l\longrightarrow\infty$.

A final remark concerning vector fields is to note that the gauge field $\vec{A}^F$ has three components and hence an extra condition is needed in order to project this gauge field onto two dimensions. One adopts here the prescription of [16], i.e. we impose on the gauge filed $\vec{A}^F$ the gauge-covariant condition

$$D^F_aD^F_a=l(l+1)\,. \tag{3.2}$$

This constraint reads explicitely $\{n^F_a,A^F_a\}=-(\vec{A}^F)^2/\sqrt{l(l+1)}$, and thus it is not difficult to check that in the continuum limit $l\longrightarrow\infty$ the normal component of the gauge field is zero, i.e. $\phi\equiv\vec{n}.\vec{A}=0$.

3.2 Fermion doubling

In close analogy with the free fermion action on ordinary ${\bf S}^2$, free fermion action on fuzzy ${\bf S}^2$ is defined by

$$S_F=\frac{1}{2l+1}\,{\rm Tr}_l\,\bar{\psi}_F\mathcal{D}_F\psi_F\,,\qquad \mathcal{D}_F=\sigma_a[L_a,\ldots]+1\,. \tag{3.3}$$

$\mathcal{D}_F$ is precisely the Dirac operator on fuzzy ${\bf S}^2$ [3]–[6], and σ_i's are Pauli matrices. The fuzzy spinor ψ_F is an element of the projective module $Mat_{2l+1}\otimes{\bf C}^2$, it is of mass dimension $({\rm mass})^{\frac{1}{2}}$ and is such that $\bar{\psi}_F=\psi^+_F$.

The so-called Grosse-Klimčík-Prešnajder Dirac operator $\mathcal{D}_F$ on fuzzy ${\bf S}^2$ admits a chirality operator which can be seen as follows, first we rewrite $\mathcal{D}_F$ in the form [3]–[6]

$$\frac{1}{2l+1}\mathcal{D}_F=\frac{1}{2}(\Gamma^R+\Gamma^L)\,, \tag{3.4}$$

where Γ^R and Γ^L are the operators

$$\Gamma^L=\frac{1}{l+1/2}\left[\vec{\sigma}.\vec{L}^L+\frac{1}{2}\right],\qquad \Gamma^R=\frac{1}{l+1/2}\left[-\vec{\sigma}.\vec{L}^R+\frac{1}{2}\right]. \tag{3.5}$$

Γ^R is the chirality operator as was shown originally in [17], this choice is also motivated by the fact that $\Gamma^R\longrightarrow-\gamma=-\vec{\sigma}\vec{n}$, when $l\longrightarrow\infty$, $(\Gamma^R)^2=\Gamma^R$, $(\Gamma^R)^+=\Gamma^R$, and $[\Gamma^R,\phi^F]=0$ for any $\phi^F\in Mat_{2l+1}$. However this Γ^R does not exactly anticommute with the Dirac operator since

$$\mathcal{D}_F\Gamma^R+\Gamma^R\mathcal{D}_F=\frac{1}{l+1/2}\mathcal{D}_F^2\Leftrightarrow\mathcal{D}_F\Gamma^R-\Gamma^L\mathcal{D}_F=0\,. \tag{3.6}$$

The continuum limit of this equation is simply given by the canonical anticommutation relation $\{\mathcal{D},\gamma\}=0$ where the continuum Dirac operator $\mathcal{D}$ is given by $\mathcal{D}=\sigma.\mathcal{L}+1$ (we have also used the fact that $\Gamma^L \longrightarrow \gamma$ when $l\longrightarrow\infty$).

It was shown in [3, 4] that the pair (Γ^R,Γ^L) defines a chiral structure on fuzzy $\mathbf{S}^2$ which satisfies *a*) the Ginsparg-Wilson relation, *b*) is without fermion doubling and *c*) has the correct continuum limit . Indeed Γ^R and Γ^L together with the identity generate a Ginsparg-Wilson algebra where the canonical Dirac-Ginsparg-Wilson operator is defined by $\mathcal{D}_{DGW}=\Gamma^R\mathcal{D}_F$ while the lattice spacing is identified as $a=\frac{2}{2l+1}$ (see [8] and references therein for more detail). The Ginsparg-Wilson relation for zero gauge field is essentially given by equation (3.6). On the other hand the absence of fermion doubling can be seen by comparing the spectrum of $\mathcal{D}_F$ which can be easily computed to be given by

$$\mathcal{D}_F(j)=\left\{\pm\left(j+\frac{1}{2}\right),j=\frac{1}{2},\frac{3}{2},\ldots,2l-\frac{1}{2}\right\}\cup\left\{j+\frac{1}{2},j=2l+\frac{1}{2}\right\} \tag{3.7}$$

with the spectrum of the continuum Dirac operator $\mathcal{D}$ given by $\mathcal{D}(j)=\{\pm(j+\frac{1}{2}),j=\frac{1}{2},\frac{3}{2},\ldots,\infty\}$ [6]. As one can immediately notice there is no fermion doubling and the spectrum of $\mathcal{D}_F$ is simply cut-off at the top eigenvalue $j=2l+\frac{1}{2}$. The continuum limit of this chiral structure is therefore obvious by construction.

The fuzzy gauged Dirac operator is obviously defined by $\mathcal{D}_{AF}=\mathcal{D}_F+\sigma_a A_a^F$ and thus the fuzzy gauged action is given by

$$S_{AF}=\frac{1}{2l+1}\,\mathrm{Tr}_l\,\bar{\psi}_F\mathcal{D}_{AF}\psi_F\equiv\frac{1}{2l+1}\,\mathrm{Tr}_l\left[\bar{\psi}_F\sigma_a[L_a,\psi_F]+\bar{\psi}_F\psi_F+\bar{\psi}_F\sigma_a A_a^F\psi_F\right]. \tag{3.8}$$

The spinor ψ_F is assumed here to transform in the fundamental representation of the gauge group, i.e. $\psi_F\longrightarrow\psi_F^{'}=U^F\psi_F$, $\bar{\psi}_F\longrightarrow\bar{\psi}_F^{'}=\bar{\psi}_F U^{F+}$. Again it is not difficult to see that both classical actions (3.3) and (3.8) behave correctly in the continuum limit in the sense that $S_F\longrightarrow S=\int_{\mathbf{S}^2}\frac{d\Omega}{4\pi}\bar{\psi}\mathcal{D}\psi$ and $S_{AF}\longrightarrow S_A=\int_{\mathbf{S}^2}\frac{d\Omega}{4\pi}\bar{\psi}\mathcal{D}_A\psi$ when $l\longrightarrow\infty$ and where $\psi=\mathrm{Lim}_{l\longrightarrow\infty}\langle\psi^F\rangle$ and $\mathcal{D}_A=\mathcal{D}+\sigma_a A_a$.

By putting the actions (3.1) and (3.8) together we obtain the fuzzy Schwinger model on $\mathbf{S}^2_F$, namely

$$S_{\text{Schwinger}}=\frac{1}{4e^2}\frac{1}{2l+1}\,\mathrm{Tr}_l\,F_{ab}^F F_{ab}^F+\frac{1}{2l+1}\,\mathrm{Tr}_l\,\bar{\psi}_F\mathcal{D}_{AF}\psi_F\,, \tag{3.9}$$

where the gauge field $\vec{A}^F$ is also assumed to satisfy the constraint (3.2).

4. Quantum chiral symmetry on $\mathbf{S}^2_F$

The quantum theory of interest is defined through the following path integral

$$\int\mathcal{D}A_i^F\int\mathcal{D}\psi_F\mathcal{D}\bar{\psi}_F e^{-S_{\text{Schwinger}}}\,. \tag{4.1}$$

The anomaly arises generally from the non-invariance of the fermionic measure under chiral transformations [18] and hence we will focus first on this measure and show explicitly that for all finite approximations of the noncommutative Schwinger model on $\mathbf{S}_F^2$ this measure is in fact exactly invariant. Indeed we will show shortly that the U(1) "fuzzy"axial anomaly on $\mathbf{S}_F^2$ comes entirely from the non-invariance of the action due to edge effect.

In a matrix model such as (4.1) manipulations on the quantum measure have a precise meaning. Indeed and by following [18] we first expand the fuzzy spinors ψ_F, $\bar{\psi}_F$ in terms of the eigentensors $\phi(\mu, A)$ of the Dirac operator $\mathcal{D}_{AF}$, write $\psi_F = \sum_\mu \theta_\mu \phi(\mu, A)$, $\bar{\psi}_F = \sum_\mu \bar{\theta}_\mu \phi^+(\mu, A)$ where θ_μ's, $\bar{\theta}_\mu$'s are independent sets of grassmanian variables, and $\phi(\mu, A)$'s are defined by $\mathcal{D}_{AF}\phi(\mu, A) = \lambda_\mu(A)\phi(\mu, A)$, and normalized such that

$$\frac{1}{2l+1}\,\mathrm{Tr}_l\,\phi^+(\mu, A)\phi(\nu, A) = \delta_{\mu\nu}\,. \tag{4.2}$$

For example for zero fuzzy gauge fields μ stands for j, k and m which are the eigenvalues of $\vec{J}^2 = (\vec{K} + \frac{\vec{\sigma}}{2})^2$, $\vec{K}^2 = (\vec{L}^L - \vec{L}^R)^2$ and J_3 respectively.

From the action S_{AF} and the identity (4.2) one can compute the propagator $\langle\theta_\mu\bar{\theta}_\nu\rangle = \delta_{\mu\nu}/\lambda_\mu(A)$ or equivalently

$$\langle\psi_{F\alpha}^{AB}\bar{\psi}_{F\beta}^{CD}\rangle_{ev} = \sum_\mu \frac{1}{\lambda_\mu(A)}\phi_\alpha^{AB}(\mu, A)\phi_\beta^{+CD}(\mu, A)\equiv(2l+1)\left(\frac{1}{\mathcal{D}_{AF}}\right)_{\alpha\beta}^{AB,DC}. \tag{4.3}$$

$\langle\rangle_{ev}$ stands for expectation value. We are assuming no monopole configurations and thus the inverse of the gauged Dirac operator is always well defined. In above we have also used the fact that because the Dirac operator $\mathcal{D}_{AF}$ is self-adjoint on $Mat_{2l+1}\otimes\mathbf{C}^2$, the states $\phi(\mu, A)$'s must form a complete set, viz

$$\frac{1}{2l+1}\sum_\mu \phi_\alpha^{AB}(\mu, A)\phi_\beta^{+CD}(\mu, A) = \delta_{\alpha\beta}\delta^{AD}\delta^{BC}\,. \tag{4.4}$$

The Dirac operator $(\mathcal{D}_{AF})_{\alpha\beta}$ and the propagator $(\mathcal{D}_{AF}^{-1})_{\alpha\beta}$ carry 4 indices because they can act on matrices of Mat_{2l+1} either from the left or from the right.

4.1 The Dirac-Ginsparg-Wilson relation on $\mathbf{S}_F^2$

We will now undertake the task of deriving the Dirac-Ginsparg-Wilson relation on $\mathbf{S}_F^2$ in the presence of a gauge field. As it turns out this relation contains exactly a fuzzy anomaly which will become a local anomaly in the limit (in the quantum theory). We first start with the free theory and rewrite the Ginsparg-Wison relation (3.6) in the form $(\Gamma^R - \Gamma^L)\mathcal{D}_F + \mathcal{D}_F(\Gamma^R - \Gamma^L) = 0$.This means that in the absence of gauge fields we must have $\mathrm{tr}[\Gamma^R - \Gamma^L] = 0$ where the trace is taken in the space of spinors, in other words the anomaly vanishes. However if we include the gauge field through the gauge-covariant Dirac operator $\mathcal{D}_{AF}$ which can also be written in the form

$$\frac{1}{2l+1}\mathcal{D}_{AF} = \frac{1}{2}(\Gamma^R + \hat{\Gamma}^L)\,, \qquad \hat{\Gamma}^L = \frac{1}{l+\frac{1}{2}}\left[\sigma_a D_a^F + \frac{1}{2}\right]. \tag{4.5}$$

Then we can compute instead the gauge-covariant anticommutation relation

$$\begin{aligned}\{\Gamma^R - \hat{\Gamma}^L, \mathcal{D}_{AF}\} &= -\frac{4}{2l+1}\Big(F^F + (D_a^F)^2 - l(l+1)\Big)\\ F^F &= \frac{i}{2}\epsilon_{abc}\sigma_c F_{ab}^F\,.\end{aligned} \tag{4.6}$$

The continuum limit of this equation is $\{\gamma, \mathcal{D}_A\} = 2\phi = 0$ where ϕ is the normal component of the gauge field on $\mathbf{S}^2$ which is zero by the continuum limit of (3.2). In other words, in the continuum interacting theory one might be tempted to conclude that $\mathrm{tr}\,\gamma = 0$ which we know is wrong in the presence of gauge fields. Noncommutative geometry, as it is already obvious from equation (4.6), already gives us the structure of the chiral anomaly, indeed by using the constraint (3.2) one can put (4.6) in the equivalent form

$$\mathcal{D}_{AF}\Gamma^R - \hat{\Gamma}^L\mathcal{D}_{AF} = -\frac{2}{2l+1}F^F\,. \tag{4.7}$$

We recognize immediately the left hand side as a fuzzy anomaly since it vanishes in the limit $A^F \longrightarrow 0$ where (4.7) reduces to (3.6). We will now show that this corresponds indeed to the actual global $U_A(1)$ fuzzy anomaly on $\mathbf{S}^2_F$.

4.2 Gauge-covariant axial current

Now we need to define chiral symmetry on the fuzzy sphere which must also be consistent with gauge invariance. The minimal prescription as it turns out is given by

$$\psi_F \longrightarrow \psi_F^{'} = \psi_F + (\Gamma^R\lambda^F\psi_F) + O(\lambda^2)\,, \qquad \bar{\psi}_F \longrightarrow \bar{\psi}_F^{'} = \bar{\psi}_F - (\bar{\psi}_F\lambda^F\hat{\Gamma}^L) + O(\lambda^2)\,, \tag{4.8}$$

where $\hat{\Gamma}^L$ is the $A-$dependent chirality-like operator defined in (4.5). These transformations are of course motivated by (4.7). Indeed one can check that (4.8) reduces in the limit to the usual chiral transformations yet it guarantees gauge invariance in the noncommutative fuzzy setting since both $\hat{\Gamma}^L$ and the chiral parameter λ^F transform covariantly as $U^F\hat{\Gamma}^LU^{F+}$ and $U^F\lambda^FU^{F+}$ respectively under gauge transformations. Naive fuzzy chiral transformations which would be again given by (4.8) but with Γ^L instead of $\hat{\Gamma}^L$ are in fact inconsistent with gauge symmetry as one might easily convince ourselves, whereas in the case of (4.8) if one gauge transform ψ_F and $\bar{\psi}_F$ by a unitary transformation U^F their chiral transform $\psi_F^{'}$ and $\bar{\psi}_F^{'}$ will also rotate by the same gauge transformation U^F.

The change of the action under these fuzzy chiral transformations (4.8) is given by

$$\int \mathcal{D}\psi_F{}^{'}\mathcal{D}\bar{\psi}_F^{'}e^{-S_{AF}^{'}} = \int \mathcal{D}\psi_F\mathcal{D}\bar{\psi}_F e^{S_{\theta F}}e^{-S_{AF}-\Delta S_{AF}}\,, \tag{4.9}$$

with

$$\begin{aligned}\Delta S_{AF} &= -\frac{1}{2l+1}\,\mathrm{Tr}_l\,\lambda^F[D_a^F, \mathcal{J}_a^5] - \frac{i}{(2l+1)^2}\epsilon_{abc}\,\mathrm{Tr}_l\,\bar{\psi}_F\lambda^F\sigma_cF_{ab}^F\psi_F \\ \mathcal{J}_a^5 &= \bar{\psi}_F\sigma_a\Gamma^R\psi_F - [(\bar{\psi}_F\sigma_a)_\alpha, (\Gamma^R\psi_F)_\alpha]\,.\end{aligned} \tag{4.10}$$

An immediate remark is that the change in the action is not simply a total covariant divergence but there is an extra piece which vanishes (as it stands) only in the continuum limit, i.e. we have a gauge-invariant edge effect. It is clear that the source of this edge effect is the RHS of the Ginsparg-Wilson relation (4.7).

The theta term in the path integral (4.9) is also gauge-invariant and it is given by

$$S_{\theta F} = -\frac{1}{2l+1}\,\mathrm{Tr}_l\sum_\mu \phi^+(\mu, A)\lambda^F(\Gamma^R - \hat{\Gamma}^L)\phi(\mu, A)\,. \tag{4.11}$$

It is however an identity easy to check that this term is identically zero, and hence it seems that the chiral WT identity $\Delta S_{GF} = S_{\theta F}$ is not anomalous due to the non-invariance of the measure under chiral transformations but rather anomalous due to edge effects (the second term of the first equation of (4.10)). We write this WT identity $\Delta S_{GF} = 0$ in the form

$$\langle [D_a^F, \mathcal{J}_a^5] \rangle_{ev} = \frac{i}{2l+1} \epsilon_{abc} \langle (\sigma_c)_{\alpha\beta} F_{ab}^F \psi_{F\beta} \bar{\psi}_{F\alpha} \rangle_{ev} . \tag{4.12}$$

Using the propagator (4.3) we find the anomaly

$$[D_a^F, \langle \mathcal{J}_a^5 \rangle]^{CB} = i\epsilon_{abc} (F_{ab}^F)^{CD} \, \mathrm{tr}_2 \, \sigma_c (D_{AF}^{-1})^{DA,BA} . \tag{4.13}$$

This is the "fuzzy" form of the global $U_A(1)$ axial anomaly on $\mathbf{S}_F^2$. The inetgrated form of this anomaly is clearly given by

$$\mathcal{A}_{\theta F} = -\frac{i}{2l+1} \epsilon_{abc} (\lambda^F)^{BC} (F_{ab}^F)^{CD} \, \mathrm{tr}_2 \, \sigma_c (\mathcal{D}_{AF}^{-1})^{DA,BA} . \tag{4.14}$$

The central claim of this article is that higher modes of the fuzzy sphere are essentially the source of the anomaly. In fact these top modes are the source of the edge effects we saw in (4.7) and in (4.10) which effectively yielded the non-vanishing anomaly (4.13). Remark that if we make the top modes larger, i.e. $l \longrightarrow \infty$, these effects in (4.7) and (4.10) become smaller while the fuzzy anomaly (4.13) remains non-zero. In the strict limit $l \longrightarrow \infty$ these edge effects in (4.7) and (4.10) completely disappear while the anomaly (4.13) survives (see below for the explicit proof). This is the origin of the anomaly in this context.

5. The continuum limit

5.1 Gauge covariant expansion on $\mathbf{S}_F^2$

From equation (4.14) one can immediately notice that the computation of the exact Dirac propagator $(\mathcal{D}_{AF})^{-1}$ is required in order to obtain a closed formula of the fuzzy anomaly and its continuum limit. Towards this end the best approach is as usual to expand the above propagator in a gauge covariant manner and then calculate the anomaly. In fact there exists a gauge covariant expansion on the noncommutative fuzzy sphere which is not available in the continuum setting and which yields a non-perturbative result in the limit. It is precisely in this sense that the fuzzy sphere is said to be a gauge-invariant, chiral-invariant regularization of the continuum physics. This is in contrast with other approximation schemes in which gauge covariant expansions are so often absent.

This gauge covariant expansion can be motivated as follows. Starting with zero gauge field one can see that the free Dirac propagator $1/\mathcal{D}_F$ admits the expansion

$$\frac{1}{\mathcal{D}_F} = \frac{a\Gamma^L}{2} + \frac{1}{\mathcal{D}_a} , \qquad \text{with} \qquad \frac{1}{\mathcal{D}_a} = ib \frac{1}{|\mathcal{D}_F|^2} \mathcal{D}_F^{'} \Gamma^L . \tag{5.1}$$

a is the lattice spacing introduced before, i.e. $a = \frac{2}{2l+1}$, $b = \sqrt{1 - a^2/4}$, and $\mathcal{D}_F^{'}$ is the Watamura Dirac operator given by $\mathcal{D}_F^{'} = \epsilon_{abc} \sigma_a n_b^F L_c^R$ [3, 17].

Next we remark that the "Dirac" operator $\mathcal{D}_a$ defined in (5.1) is such that $\mathcal{D}_a^{-2} = \mathcal{D}_F^{-2} - a^2/4$. In other words on the top modes $j = 2l + 1/2$, $(\mathcal{D}_a)^2$ is strictly infinite whereas for the other large j's the eigenvalues of $(\mathcal{D}_a)^2$ are quite large. On the IR modes the spectrum of $(\mathcal{D}_a)^2$ is essentially equal to that of $\mathcal{D}_F^2$. This means in particular that as far as the second term $\mathcal{D}_a^{-1}$ in (5.1) is concerned modes with large j's do not effectively propagate.

We will show (after we also take the gauge field into account) that the contribution of these UV modes to the anomaly translates in the continuum limit into the contribution of contact terms whereas the contribution of the IR modes vanishes identically. As it turns out contact terms yield only a toplogical Chern-Simons lagrangian which in two dimensions vanishes identically. This also reflects in a sense the fact that the above UV modes are suppressed in the propagator $(\mathcal{D}_a)^{-1}$. On the other hand although the first term $a\Gamma^L/2$ in (5.1) vanishes as it stands in the continuum limit, its contribution in that limit is not zero and is exactly given by the canonical theta term. Putting these facts together we conclude that the anomaly emerges essentially from the UV region of the spectrum as expected. We now give a rigorous proof of this result.

In the presence of gauge fields the free expansion (5.1) becomes a covariant expansion given by

$$\frac{1}{\mathcal{D}_{AF}} = \frac{a\hat{\Gamma}^L}{2} - \frac{a^3}{2}\frac{F^F}{1+a^2F^F}\hat{\Gamma}^L + \frac{1}{\mathcal{D}_{Aa}}, \tag{5.2}$$

with

$$\frac{1}{\mathcal{D}_{Aa}} = \frac{1}{\mathcal{D}_{AF}^2}\left[ib\mathcal{D}'_{AF} + \frac{a}{2}F^F\right]\frac{1}{1+a^2F^F}\hat{\Gamma}^L. \tag{5.3}$$

It is not difficult to see that each term in this expansion is exactly covariant under gauge transformations. It is also obvious that (5.2) reduces in the limit $A_a^F \longrightarrow 0$ to (5.1). Furthermore in the continuum limit $l\longrightarrow\infty$, (5.2) reduces to $\frac{1}{\mathcal{D}_A} = \frac{1}{\mathcal{D}_A^2}(i\mathcal{D}'_A\gamma)$ which is actually an identity since $\mathcal{D}_A = i\mathcal{D}'_A\gamma$ and thus the expansion (5.2) is simply not available to us in the continuum.

In order to compute the contribution of $(\mathcal{D}_{Aa})^{-1}$ it is obvious that one needs also an expansion of $\frac{1}{\mathcal{D}_{AF}^2}$. To this end we recall first that the square of the Dirac operator $\mathcal{D}_{AF}$ is given by $\mathcal{D}_{AF}^2 = \mathcal{D}_{AF} + (L_a - L_a^R + A_a^F)^2 + \frac{i}{2}\epsilon_{abc}\sigma_c F_{ab}^F$. By using the constraint (3.2) we can write this square in the form

$$\mathcal{D}_{AF}^2 = 2l(l+1)P_{AF}\left[1 - \frac{1}{P_{AF}}v_a^F n_a^R\right], \tag{5.4}$$

where n_a^R's are the fuzzy coordinates which act from the right, i.e. $n_a^R = \frac{L_a^R}{\sqrt{l(l+1)}}$ and where

$$P_{AF} = 1 + \frac{1 + F^F + \sqrt{l(l+1)}\sigma x^F}{2l(l+1)}, \qquad v_a^F = x_a^F + \frac{\sigma_a}{2\sqrt{l(l+1)}}. \tag{5.5}$$

It is clear that the only bit in $\mathcal{D}_{AF}^2$ which acts on the right is $\vec{n}^R$ and that P_{AF} acts entirely on the left . This means that the operator $(P_{AF})_{\alpha\beta}$ can now be treated as a matrix in Mat_{2l+1}. We think now of P_{AF} as a propagator and of $n_a^R v_a^F$ as a vertex and write the expansion

$$\frac{1}{\mathcal{D}_{AF}^2}=\frac{1}{2l(l+1)}\sum_{N=0}^{\infty}\left(\frac{1}{P_{AF}}v_{a_1}^F\frac{1}{P_{AF}}v_{a_2}^F\dots\frac{1}{P_{AF}}v_{a_N}^F\frac{1}{P_{AF}}\right)\left(n_{a_1}^R n_{a_2}^R\dots n_{a_N}^R\right). \tag{5.6}$$

The meaning of this expansion will only be clear in the continuum limit which we will take shortly. The propagator $1/P_{AF}$ acts now from the left and thus it can also be thought of as a matrix in $Mat_{2l+1}\otimes Mat_2$ rather than as an operator. It is given by

$$\frac{1}{P_{AF}}=\left(1-\frac{a}{2}\hat{\Gamma}^L-\frac{a^2}{2}F^F\right)\frac{1}{1-\frac{a^3}{4-a^2}\{\hat{\Gamma}^L,F^F\}-\frac{a^4}{4-a^2}(F^F)^2}. \tag{5.7}$$

Therefore given any two operators $(X_a^L)_{\alpha\beta}$ and Y_b^R which act from the left and from the right respectively, i.e. $(X_a^L)_{\alpha\beta}f\equiv(X_a)_{\alpha\beta}f$ and $Y_b^Rf\equiv fY_b$ for any $f\in Mat_{2l+1}$, we can compute

$$\begin{aligned}\left(\frac{1}{\mathcal{D}_{AF}^2}X_a^LY_b^R\right)_{\alpha\beta}^{AB,CD}&=\frac{1}{2l(l+1)}\sum_{N=0}^{\infty}\left(\frac{1}{P_{AF}}v_{a_1}^F\frac{1}{P_{AF}}v_{a_2}^F\dots\frac{1}{P_{AF}}v_{a_N}^F\frac{1}{P_{AF}}X_a\right)_{\alpha\beta}^{AC}\times\\&\times\left(Y_bn_{a_N}^F\dots n_{a_2}^Fn_{a_1}^F\right)^{DB}.\end{aligned} \tag{5.8}$$

To summarize, the gauge covariant expansion of the propagator $\frac{1}{\mathcal{D}_{AF}}$ is defined by equations (5.2) , (5.3), (5.6) and (5.7). Using these equations together with the above definition one obtains an explicit formula for the exact quark propagator on $\mathbf{S}_F^2$, viz

$$\left(\frac{1}{\mathcal{D}_{AF}}\right)_{\alpha\beta}^{AB,CD}=\left(\frac{a\hat{\Gamma}^L}{2}-\frac{a^3}{2}\frac{F^F}{1+a^2F^F}\hat{\Gamma}^L\right)_{\alpha\beta}^{AC}\delta^{BD}+\left(\frac{1}{\mathcal{D}_{Aa}}\right)_{\alpha\beta}^{AB,CD}, \tag{5.9}$$

with

$$\begin{aligned}\left(\frac{1}{\mathcal{D}_{Aa}}\right)_{\alpha\beta}^{AB,CD}&=\frac{ib\epsilon_{abc}}{2l(l+1)}\sum_{N=0}^{\infty}\left(\frac{1}{P_{AF}}v_{a_1}^F\frac{1}{P_{AF}}v_{a_2}^F\dots\frac{1}{P_{AF}}v_{a_N}^F\frac{1}{P_{AF}}\sigma_ax_b^F\frac{1}{1+a^2F^F}\hat{\Gamma}^L\right)_{\alpha\beta}^{AC}\times\\&\times\left(L_cn_{a_N}^F\dots n_{a_2}^Fn_{a_1}^F\right)^{DB}+\frac{a}{4l(l+1)}\times\\&\times\sum_{N=0}^{\infty}\left(\frac{1}{P_{AF}}v_{a_1}^F\frac{1}{P_{AF}}v_{a_2}^F\dots\frac{1}{P_{AF}}v_{a_N}^F\frac{1}{P_{AF}}\frac{F^F}{1+a^2F^F}\hat{\Gamma}^L\right)_{\alpha\beta}^{AC}\times\\&\times\left(n_{a_N}^F\dots n_{a_2}^Fn_{a_1}^F\right)^{DB}.\end{aligned} \tag{5.10}$$

As we have explained this gauge covariant expansion does not exist on the continuum sphere. It will be used here to derive the fuzzy axial anomaly and its continuum limit the local axial anomaly.

5.2 Local axial anomaly

This expansion isolates in fact (in a gauge covariant fashion) the anomalous bit in the fermion propagator. More precisely, although the first term in (5.9) vanishes as it stands in the large l limit, its contribution (i.e. its trace) gives exactly the canonical anomaly. Indeed we can easily compute

$$\begin{aligned}\delta_1\mathcal{A}_{\theta F} &\equiv -\frac{i}{2l+1}\epsilon_{abc}(\lambda^F F^F_{ab})^{BD}\left[\mathrm{tr}_2\,\sigma_c\left(\frac{a\hat{\Gamma}^L}{2}\right)^{DA,BA}+\mathrm{tr}_2\,\sigma_c\left(-\frac{a^3}{2}\frac{F^F}{1+a^2F^F}\hat{\Gamma}^L\right)^{DA,BA}\right]\\ &= -2b\frac{i\epsilon_{abc}}{2l+1}\,\mathrm{Tr}_l\,\lambda^F F^F_{ab}\delta_1 x^F_c\,, \qquad (5.11)\end{aligned}$$

where

$$\delta_1 x^F_c = x^F_c - \frac{a^2}{2b}\,\mathrm{tr}_2\left(\sigma_c\frac{F^F}{1+a^2F^F}\hat{\Gamma}^L\right). \qquad (5.12)$$

This gauge-covariant vector $\vec{\delta}_1 x^F$ in the continuum limit becomes exactly the unit vector $\vec{n}$ on the 2−dimensional sphere and hence the fuzzy anomaly (5.11) reduces in that limit to the theta term on $\mathbf{S}^2$, in other words

$$-2b\frac{i\epsilon_{abc}}{2l+1}\,\mathrm{Tr}_l\,\lambda^F F^F_{ab}\delta_1 x^F_c \longrightarrow -2i\epsilon_{abc}\int_{\mathbf{S}^2}\frac{d\Omega}{4\pi}\lambda(\vec{n})F_{ab}(\vec{n})n_c\,, \qquad \text{when} \quad l\longrightarrow\infty\,. \qquad (5.13)$$

For example by using the star product on $\mathbf{S}^2_F$ one can see that in the continuum limit $l\longrightarrow\infty$ where $a\longrightarrow 0$ and $b\longrightarrow 1$ we have $\delta_1 x^F_a\longrightarrow n_a$, $F^F_{ab}\longrightarrow F_{ab}$, $\lambda^F\longrightarrow\lambda$ while the trace $\frac{1}{2l+1}\mathrm{Tr}_l$ becomes the integral $\int_{\mathbf{S}^2}\frac{d\Omega}{4\pi}$.

5.3 The Chern-Simons action

From equation (5.3) one can immediately read the remaining two extra corrections corresponding to the propagator $1/\mathcal{D}_{Aa}$. We find

$$\begin{aligned}\delta_2\mathcal{A}_{\theta F} &= -2b\frac{i\epsilon_{abc}}{2l+1}\,\mathrm{Tr}_l\,\lambda^F F^F_{ab}\delta_2 x^F_c \qquad (5.14)\\ \delta_2 x^F_c &= \sum_{N=1}\left(\frac{1}{2l+1}\,\mathrm{Tr}_l(n^F_{a_N}\ldots n^F_{a_1}n^F_r)\right)\left(\mathrm{tr}_2\left(\frac{1}{P_{AF}}v^F_{a_1}\frac{1}{P_{AF}}v^F_{a_2}\cdots\frac{1}{P_{AF}}v^F_{a_N}\delta^F_{rc}\right)\right)\end{aligned}$$

and

$$\begin{aligned}\delta_3\mathcal{A}_{\theta F} &= -2b\frac{i\epsilon_{abc}}{2l+1}\,\mathrm{Tr}_l\,\lambda^F F^F_{ab}\delta_3 x^F_c \qquad (5.15)\\ \delta_3 x^F_c &= \sum_{N=0}\left(\frac{1}{2l+1}\,\mathrm{Tr}_l(n^F_{a_N}\ldots n^F_{a_1})\right)\left(\mathrm{tr}_2\left(\frac{1}{P_{AF}}v^F_{a_1}\frac{1}{P_{AF}}v^F_{a_2}\cdots\frac{1}{P_{AF}}v^F_{a_N}\delta^F_c\right)\right),\end{aligned}$$

where δ^F_{rc} and δ^F_c stand for

$$\delta^F_{rc} = \frac{i\epsilon_{rpq}}{2b}\frac{1}{P_{AF}}\sigma_p x^F_q\frac{1}{1+a^2F^F}\hat{\Gamma}^L\sigma_c\,, \qquad \delta^F_c = \frac{a^2}{4b^3}\frac{1}{P_{AF}}F^F\frac{1}{1+a^2F^F}\hat{\Gamma}^L\sigma_c\,. \qquad (5.16)$$

Remark that the components of the two vectors $\vec{\delta}_2 x^F$ and $\vec{\delta}_3 x^F$ are matrices in Mat_{2l+1} which are covariant under gauge transformations. We can now immediately write down the continuum limit of the above expressions. We already know that in this limit $a\longrightarrow 0$, $b\longrightarrow 1$,

$\frac{1}{2l+1}\,\mathrm{Tr}_l \longrightarrow \int_{\mathbf{S}^2}$, $n_a^F \longrightarrow n_a$, $x_a^F \longrightarrow n_a$, $F_{ab}^F \longrightarrow F_{ab}$, $\lambda^F \longrightarrow \lambda$ and $\hat{\Gamma}^L \longrightarrow \gamma$. From equation (5.5) we also see that in this limit $P_{AF} \longrightarrow 1$ and $v_a^F \longrightarrow n_a$. For example the continuum limit of (5.14) is given by

$$\begin{aligned} \delta_2\mathcal{A}_\theta &= -2i\epsilon_{abc}\int_{\mathbf{S}^2}\frac{d\Omega}{4\pi}\lambda(\vec{n})F_{ab}(\vec{n})\delta_2 x_c(\vec{n}) \\ \delta_2 x_c(\vec{n}) &= \frac{i}{2}\int_{\mathbf{S}^2}\frac{d\Omega^{'}}{4\pi}(\vec{n}\times\vec{n}^{'})_p tr_2\Big(\sigma_c\sum_{N=1}(\vec{n}.\vec{n}^{'})^N\sigma_p\gamma(\vec{n})\Big)\,. \end{aligned} \tag{5.17}$$

It is easily seen that all the expected continuum divergence arises from the series $\sum_{N=1}(\vec{n}.\vec{n}^{'})^N$ when $\vec{n}=\vec{n}^{'}$, i.e. from contact terms. Thus we need first to separate contact terms as follows

$$\delta_2 x_c(\vec{n}) = \frac{i}{2}\Big[(\vec{n}\times\vec{n}^{'})_p tr_2\Big(\sigma_c\sum_{N=1}(\vec{n}.\vec{n}^{'})^N\sigma_p\gamma(\vec{n})\Big)\Big]_{\vec{n}^{'}=\vec{n}} + \delta_2 x_c(\vec{n})|_{\mathrm{nct}}\,, \tag{5.18}$$

where

$$\delta_2 x_c(\vec{n})|_{nct} = \frac{i}{2}\int_{\vec{n}^{'}\neq\vec{n}}\frac{d\Omega^{'}}{4\pi}(\vec{n}\times\vec{n}^{'})_p\,\mathrm{tr}_2\,\Big(\sigma_c\sum_{N=1}(\vec{n}.\vec{n}^{'})^N\sigma_p\gamma(\vec{n})\Big)\,. \tag{5.19}$$

It is clear that in $\delta_2 x_c^F(\vec{n})|_{nct}$ we have $|\vec{n}.\vec{n}^{'}| = |cos\theta| < 1$ where θ is the angle between $\vec{n}$ and $\vec{n}^{'}$ and thus the series $\sum_{N=1}(\vec{n}.\vec{n}^{'})^N$ is now convergent. We can therefore simply substitute the result $\sum_{N=1}(\vec{n}.\vec{n}^{'})^N = \frac{\vec{n}.\vec{n}^{'}}{1-\vec{n}.\vec{n}^{'}}$, and thus obtain

$$\delta_2 x_c(\vec{n})|_{nct} = \int_{\vec{n}^{'}\neq\vec{n}}\frac{d\Omega^{'}}{4\pi}\Big(n_c(\vec{n}.\vec{n}^{'})-n_c^{'}\Big)\frac{\vec{n}.\vec{n}^{'}}{1-\vec{n}.\vec{n}^{'}} = (n_c n_d-\delta_{cd})n_e\int_{\mathbf{S}^2}\frac{d\Omega^{'}}{4\pi}\frac{n_d^{'}n_e^{'}}{1+\epsilon-\vec{n}.\vec{n}^{'}}\,, \tag{5.20}$$

where we have replaced $\int_{\vec{n}^{'}\neq\vec{n}}\frac{d\Omega^{'}}{4\pi}$ by the intgeral over the full sphere $\int_{\mathbf{S}^2}\frac{d\Omega^{'}}{4\pi}$ with the prescription $\epsilon\rangle 0$ so that the contribution of the terms $\vec{n}^{'}=\vec{n}$ is kept equal to zero. Remark that $n_c\delta_2 x_c(\vec{n})|_{nct}=0$ and hence the contribution of the above non-contact terms is identically zero since in the continuum we also have $n_a A_a = 0$, $\epsilon_{abc}F_{ab} = 2in_c\partial_b(A_b)$.

For contact terms the series $\sum_{N=1}(\vec{n}.\vec{n}^{'})^N$ is obviously divergent and thus the first term of equation (5.18) is not well defined. Thus it is natural to go back to the finite expression (5.14) and think of it now as a gauge-invariant regularization of contact terms with a cut-off l provided here by the fuzzy sphere. Since we are interested in the continuum theory the cut-off l is effectively large and thus one may retain only the first few corrections to the continuum theory. As it turns out the contribution of contact terms is regularized completely and in a gauge-invariant manner if we only keep $O(\frac{1}{l})$ corrections. To see how all this works explicitly, we use the star product on $\mathbf{S}_F^2$ and replace $\delta_2 x_c^F$ by its image $\langle\delta_2 x_c^F\rangle(\vec{n})$ via the coherent states, namely

$$\begin{aligned} \langle\delta_2 x_c^F\rangle(\vec{n}) = \sum_{N=1}&\Big(\int_{\mathbf{S}^2}\frac{d\Omega^{'}}{4\pi}\langle n_{a_N}^F\rangle*\cdots*\langle n_{a_1}^F\rangle*\langle n_r^F\rangle(\vec{n}^{'})\Big)\times \\ &\times\Big(\mathrm{tr}_2\Big\langle\frac{1}{P_{AF}}\Big\rangle*\langle v_{a_1}^F\rangle*\Big\langle\frac{1}{P_{AF}}\Big\rangle*\langle v_{a_2}^F\rangle*\cdots*\Big\langle\frac{1}{P_{AF}}\Big\rangle*\langle v_{a_N}^F\rangle*\langle\delta_{rc}^F\rangle(\vec{n})\Big)\,. \end{aligned}$$

Since non-contact terms are regular in the limit one can still treat them in the same way as before and hence their contribution is zero. This means that $\langle\delta_2 x_c^F\rangle(\vec{n})$ is dominated by contact terms, namely

$$\langle\delta_2 x_c^F\rangle(\vec{n}) = \sum_{N=1}\left(\langle n_{a_N}^F\rangle * \cdots * \langle n_{a_1}^F\rangle * \langle n_r^F\rangle(\vec{n})\right) \times$$
$$\times\left(\mathrm{tr}_2\left\langle\frac{1}{P_{AF}}\right\rangle * \langle v_{a_1}^F\rangle * \left\langle\frac{1}{P_{AF}}\right\rangle * \langle v_{a_2}^F\rangle * \cdots * \left\langle\frac{1}{P_{AF}}\right\rangle * \langle v_{a_N}^F\rangle * \langle\delta_{rc}^F\rangle(\vec{n})\right).$$

From equation (2.8) we have $\langle n_a^F\rangle = n_a - \frac{1}{2l}n_a + O(1/l^2)$ and also we can compute to this order in $\frac{1}{l}$ the following quantities

$$\left\langle\frac{1}{P_{AF}}\right\rangle = 1 - \frac{1}{2l}\gamma + O\left(\frac{1}{l^2}\right)$$
$$\langle\hat{\Gamma}^L\rangle = \gamma + \frac{1}{l}\left(-\frac{1}{2}\gamma + \sigma A + \frac{1}{2}\right) + O\left(\frac{1}{l^2}\right)$$
$$\langle v_a^F\rangle = n_a + \frac{1}{l}\left(-\frac{1}{2}n_a + A_a + \frac{\sigma_a}{2}\right) + O\left(\frac{1}{l^2}\right),$$

and

$$\langle\delta_{rc}^F\rangle = \frac{i}{2}\epsilon_{rpq}\left(n_q\sigma_p\gamma\sigma_c + \frac{1}{l}\sigma_p\left(-n_q\gamma + n_q\sigma A + \frac{1}{2}n_q + \gamma A_q\right)\sigma_c - \frac{1}{2l}n_q\gamma\sigma_p\gamma\sigma_c\right) + O\left(\frac{1}{l^2}\right). \tag{5.21}$$

Then by using the star product on $\mathbf{S}_F^2$ we compute

$$\left\langle\frac{1}{P_{AF}}\right\rangle * \langle v_a^F\rangle = \left\langle\frac{1}{P_{AF}}\right\rangle\langle v_a^F\rangle + O\left(\frac{1}{l^2}\right) = n_a + \frac{1}{l}\left(-\frac{1}{2}n_a + A_a + \frac{\sigma_a}{2} - \frac{1}{2}\gamma n_a\right) + O\left(\frac{1}{l^2}\right). \tag{5.22}$$

Remark for example the continuum limit $\delta_{rc}^F \longrightarrow \delta_{rc} = \frac{i}{2}\epsilon_{rpq}\sigma_p n_q\gamma\sigma_c$ when $l\longrightarrow\infty$ as well as the limit $\langle\frac{1}{P_{AF}}\rangle * \langle v_a^F\rangle \longrightarrow n_a$ when $l\longrightarrow\infty$. Finally and by using once again the star product we compute the contribution of contact terms in the form

$$\langle\delta_2 x_c^F\rangle(\vec{n}) = \mathrm{tr}_2\left(\sum_{N=1}\left(\left\langle\frac{1}{P_{AF}}\right\rangle\langle v_a^F\rangle\langle n_a^F\rangle\right)^N\langle\delta_{rc}^F\rangle\langle n_r^F\rangle\right)$$
$$= (l+1)\left(\mathrm{tr}_2\langle\delta_{rc}^F\rangle\right)\langle n_r^F\rangle,$$

where we have also used the result

$$\sum_{N=1}\left(\left\langle\frac{1}{P_{AF}}\right\rangle\langle v_a^F\rangle\langle n_a^F\rangle\right)^N = \sum_{N=1}\left(1 - \frac{1}{l}\right)^N = l+1. \tag{5.23}$$

In other words the sum over N is linearly divergent with l and hence the remaining term $(\mathrm{tr}_2\langle\delta_{rc}^F\rangle)\langle n_r^F\rangle$ must approach in the continuum limit 0 at least as $\frac{1}{l}$ in order to reproduce a finite expression. A final calculation shows that this is indeed the case where from the equation

$$\mathrm{tr}_2\langle\delta_{rc}^F\rangle = (n_r n_c - \delta_{rc}) + \frac{i\epsilon_{rcq}n_q}{l} - \frac{1}{l}[n_r n_c - \delta_{rc} - n_c A_r - n_r A_c], \tag{5.24}$$

we deduce that $(\,\mathrm{tr}_2\langle\delta^F_{rc}\rangle)\langle n^F_r\rangle = \frac{1}{l}A_c$. The contribution of contact terms is therefore finite equal to $\langle\delta_2 x^F_c\rangle(\vec{n}) = A_c$ and as a consequence

$$\delta_2\mathcal{A}_\theta = -2i\epsilon_{abc}\int_{\mathbf{S}^2}\frac{d\Omega}{4\pi}\lambda(\vec{n})F_{ab}(\vec{n})A_c(\vec{n})\,. \tag{5.25}$$

This is exactly the topological Chern-Simons action in two dimensions and thus it vanishes identically as one might easily check (in fact it vanishes precisely because of the constraint (3.2)). Its emergence from contact terms of the propagator $\frac{1}{\mathcal{D}_{aA}}$ is intimately related to the fact that the top modes of the Dirac operator $\mathcal{D}_{Aa}$ as opposed to those of the exact Diarc operator $\mathcal{D}_{AF}$ are very large.

Similarly to above the last correction to the anomaly given by equation (5.15) has the limit

$$\delta_3\mathcal{A}_\theta = -2i\epsilon_{abc}\int_{\mathbf{S}^2}\frac{d\Omega}{4\pi}\lambda(\vec{n})F_{ab}(\vec{n})\delta_3 x_c(\vec{n})\,,\qquad \delta_3 x_c(\vec{n}) = \int_{\mathbf{S}^2}\frac{d\Omega^{'}}{4\pi}\,\mathrm{tr}_2\left(\sum_{N=0}(\vec{n}.\vec{n}^{'})^N\delta_c(\vec{n})\right). \tag{5.26}$$

The vector $\vec{\delta}(\vec{n})$ in above is in fact identically zero. Again for non-contact terms the series $\sum_{N=0}(\vec{n}.\vec{n}^{'})^N$ converges and thus the corresponding contribution clearly vanishes. For contact terms we have to regularize as before but since now the vector $\langle\vec{\delta}^F\rangle(\vec{n})$ vanishes as $\frac{1}{l^2}$ and not as $\frac{1}{l}$ we immediately conclude that the contribution of contact terms is zero in this case and thus we obatin the result $\delta_3\mathcal{A}_\theta = 0$.

6. Conclusion

The fuzzy sphere very much like lattice gauge theory is meant to be a nonperturbative regularization of quantum field theory in two dimensions. The analogy with the continuum sphere is rather straightforward. For example $\mathbf{S}^2$ is generally used as an IR regulator, which keeps track of the topology of the problem, of the commutative plane $\mathbb{R}^2$. Similarly it is generally believed that the fuzzy sphere will act as an IR regulator of the noncommutative Moyal plane. The limits of interest are obviously the flattening limits $R\longrightarrow\infty$ in the case of $\mathbb{R}^2$, and $R\longrightarrow\infty$, $l\longrightarrow\infty$ keeping the ratio $R^2/l = \theta^2$ fixed in the case of $\mathbb{R}^2_\theta$. However and as it turns out this procedure is not at all enough to remove the UV-IR mixing in perturbation theory for scalar models on $\mathbf{S}^2_F$ [19]. In fact perturbation theory of the pure gauge sector itself has a UV-IR mixing as we will be reporting elsewhere. The conclusion we draw is therefore to say that the UV-IR mixing is a generic phenomena in perturbation theory and there remains only to explore other approach to noncommutative quantum field theory and check there whether this mixing is indeed a genuine property of NQFT or merely an artifact of perturbation theory.

In this article we have set $R = 1$ for simplicity then studied as an example the axial anomaly in two dimensions on $\mathbf{S}^2_F$ and its continuum limit on $\mathbf{S}^2$. The treatement of the anomaly in this article is intrinsically nonperturbative in the sense that all our results on the fuzzy sphere as well as those of the continuum limit are in fact exact.

We have shown that the fuzzy sphere is indeed a gauge-invariant as well as a chiral-invariant regulator of the Schwinger model without fermion doubling. In particular chiral

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invariance of the classical action in this model is encoded in the Ginsparg-Wilson relation (4.7) with corresponding symmetries (4.8). We immediately notice that the edge effect in the RHS of (4.7) is essentially the anomaly we want yet it vanishes as it stands in any commutative limit. However it is only under quantization that this edge effect becomes exactly the axial anomaly in two dimensions with the correct continuum limit.

We have also shown that the star product on the fuzzy sphere is indeed a UV cut-off which precisely regularizes the continuum limit. No extra regularization was required throughout and no UV-IR mixing was found. In fact we have reproduced an exact answer to all orders in perturbation theory which coincides with the canonical answer on the continuum sphere upto a topological term (This is unlike [10] in which an expansion in powers of the gauge field was involved in order to derive the same theta term). This result is in fact in total accord with the large l analysis of the pure gauge sector on the fuzzy sphere done recently in [20] in which the UV-IR mixing was also absent. In our treatement as well as that of [20] the fuzzy sphere is really only a regulator and nothing more. The only trace of the noncommutativity we found in this article is the Chern-Simons action which still vanishes because of our constraint (3.2).

In conclusion we summarize the result of this paper in three main points :

1. We showed that the Dirac-Ginsparg-Wilson relation on $\mathbf{S}^2_F$ contains already at the classical level the anomaly in the form of an edge effect which under quantization becomes precisely the "fuzzy" $\mathrm{U}(1)_A$ axial anomaly on the fuzzy sphere.

2. We derived a novel gauge-covariant expansion of the quark propagator in the form $\frac{1}{\mathcal{D}_{AF}} = \frac{a\hat{\Gamma}^L}{2} + \frac{1}{\mathcal{D}_{Aa}}$ where $a = \frac{2}{2l+1}$ is the lattice spacing on $\mathbf{S}^2_F$, $\hat{\Gamma}^L$ is the covariant noncommutative chirality and $\mathcal{D}_{Aa}$ is an effective Dirac operator which has essentially the same IR spectrum as $\mathcal{D}_{AF}$ but differes from it on the UV modes since the eigenvalues of $\mathcal{D}_{Aa}$ on the top modes are very large compared to those of $\mathcal{D}_{AF}$. Most remarkably however is the fact that both operators share the same continuum limit and thus the above covariant expansion is not available in the continuum theory.

3. The first bit in this expansion $\frac{a\hat{\Gamma}^L}{2}$ although it vanishes as it stands in the continuum limit, its contribution to the anomaly is exactly the canonical theta term. The contribution of the propagator $\frac{1}{\mathcal{D}_{Aa}}$ is on the other hand equal to the toplogical Chern-Simons action which in two dimensions is identically zero. In particular we have explicilty shown that beside the cut-off l provided by the star product of the fuzzy sphere itself there is no need to any extra regulator even while approaching the limit.

Finally we have to note that a complete extension of the above results to the case of monopoles is more or less straightforward after we identify correctly the corresponding bundle structure on $\mathbf{S}^2_F$. As it turns out this is indeed possible and thus the extension can be made without much difficulty. The relevant detail will be however reported elsewhere. The computation of the effective action of the Schwinger model on the fuzzy two-sphere, which will allow us on the other hand to probe the solvability of the model, will also be reported elsewhere.

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